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Singular Integrals on Product Spaces

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INTRODUCTION

In their well-known theory of singular integrals on $\mathbf{R}^n$, Calderón and Zygmund [2] obtained the boundedness of certain convolution operators on $\mathbf{R}^n$ which generalize the Hilbert transform on $\mathbf{R}^1$. Thus, we know that if $T(f) = f * K$ and $K(x)$ is defined on $\mathbf{R}^n$ and satisfies the analogous estimates that $1/x$ satisfies on $\mathbf{R}^1$, namely

$$|K(x)| \leqslant \frac{C}{|x|^n}, \tag{1}$$

$$\int_{\alpha < |x| < \beta} K(x)\, dx = 0, \qquad 0 < \alpha < \beta, \tag{2}$$

and

$$\int_{|x| > 2|h|} |K(x+h) - K(x)|\, dx \leqslant C, \qquad \text{for all } h \neq 0, \tag{3}$$

then T is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

Furthermore, if (3) is replaced by somewhat stronger assumptions such as $|\nabla K(x)| \leqslant C/|x|^{n+1}$, and if we define T_ε, the truncated singular integral, by $T_\varepsilon(f) = f * (K\chi_{|x| > \varepsilon})$ then one can obtain the maximal estimates

$$\|\sup_{\varepsilon > 0} |T_\varepsilon f(x)|\|_{L^p} \leqslant C_p \|f\|_{L^p}, \qquad 1 < p < \infty,$$

which establish the existence a.e. of the pointwise limit

$$\lim_{\varepsilon \to 0} T_\varepsilon f(x).$$

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Later, Hunt, Muckenhoupt, and Wheeden (see also [3]) proved weighted norm versions of the Calderón–Zygmund results, that is, estimates of the form

$$\int_{\mathbf{R}^n} |Tf|^p w\, dx \leqslant C \int_{\mathbf{R}^n} |f|^p w\, dx,$$

where w satisfies the so-called A^p condition.

Now, if we take the space $\mathbf{R}^n \times \mathbf{R}^m$ along with the *two* parameter family of dilations $(x, y) \to (\delta_1 x, \delta_2 y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $\delta_i > 0$, instead of the usual one-parameter dilations, we are led to consider operators which generalize the double Hilbert transform on $\mathbf{R}^2$, $H(f) = f * 1/xy$. The boundedness properties of H are, in general, very easy to obtain by an iteration argument. But if we consider operators $Tf = f * K$ where K is defined on $\mathbf{R}^n \times \mathbf{R}^m$ and satisfies all the analogous estimates to those satisfied by $1/xy$, but cannot be written in the form $K_1(x) \cdot K_2(y)$ then the arguments which deal with H fail.

In [6] one of us gave conditions on the kernel K analogous to (1), (2), and (3), but in the product setting, which guarantee that the operator T is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Here we want to elaborate these results and present further theorems on singular integrals in product domains which correspond to the weighted norm inequalities and the maximal inequalities for truncated singular integrals on $\mathbf{R}^n$. From this it is evident that many results for classical singular integrals can be extended to the product situation at hand.

Our assumptions on the kernel $K = K(x, y)$, $((x, y) \in \mathbf{R}^n \times \mathbf{R}^m)$, will in each case be weaker versions of the following basic assumptions, given by a "cancellation" property and a "size" property, namely

$$\int_{\alpha < |x| < \beta} K(x, y)\, dx = 0 \qquad \text{for all } 0 < \alpha < \beta, \text{ and } y \in \mathbf{R}^m,$$

and (C.0)

$$\int_{\alpha < |y| < \beta} K(x, y)\, dy = 0 \qquad \text{for all } 0 < \nu < \beta, \text{ and } x \in \mathbf{R}^n.$$

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} K(x, y) \right| \leqslant A_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-m-|\beta|}. \tag{S.0}$$

In Section 1 we deal with conditions on K (the most general we assume) that guarante the boundedness in L^2. Next, under somewhat stronger assumptions, we show the L^p boundedness in Section 2, and its weighted version in Section 3. Section 4 deals with the maximal inequalities for L^p, and in Section 5 some results for $L \log^+ L$ are presented. Section 6 will deal with a class of examples which illustrate part of the theory.

It should be pointed out that the complexity of the methods in Sections 4 and 5 is due in part to difficulties which are characteristic of the situation arising in product domains: the comparison of regularized singular integrals with their truncations, and the distinction between standard truncations and smooth truncations.

1. L^2 Estimates

Suppose $K(x, y)$ is a function on $\mathbf{R}^n \times \mathbf{R}^m$, locally integrable away from the cross $\{x=0\} \cup \{y=0\}$. Define

$$\Delta_h^1 K(x, y) = K(x+h, y) - K(x, y),$$
$$\Delta_k^2 K(x, y) = K(x, y+k) - K(x, y),$$

and

$$\Delta_{h,k}^{1,2}(K) = \Delta_h^1(\Delta_k^2(K)).$$

Let us consider the following "cancellation" and "size" properties:

$$\text{(a)} \quad \left| \iint_{\alpha_1 < |x| < \alpha_2, \beta_1 < |y| < \beta_2} K(x, y)\, dx\, dy \right| \leqslant A, \qquad \text{all } 0 < \alpha_1 < \alpha_2, \; 0 < \beta_1 < \beta_2, \tag{C.1}$$

(b) If $K_1(x) = \int_{\beta_1 < |y| < \beta_2} K(x, y)\, dy$, then

$$\int_{|x| \leqslant r} |x|\, |K_1(x)|\, dx \leqslant Ar, \qquad \text{and} \qquad \int_{|x| \geqslant 2|h|} |\Delta_h^1 K_1(x)|\, dx \leqslant A;$$

A similar condition holds with $K_2(y) = \int_{\alpha_1 < |x| < \alpha_2} K(x, y)\, dx$ replacing K_1.

$$\text{(a)} \quad \iint_{|y| \leqslant \rho, |x| \leqslant r} |x|\, |y|\, |K(x, y)|\, dx\, dy \leqslant Ar\rho, \qquad 0 < r < \infty, \; 0 < \rho < \infty, \tag{S.1}$$

$$\text{(b)} \quad \int_{|y| \leqslant \rho} \int_{|x| \geqslant 2|h|} |y|\, |\Delta_h^1 K(x, y)|\, dx\, dy \leqslant A\rho, \qquad \rho > 0,$$

with a similar condition for Δ_k^2

$$\text{(c)} \quad \iint_{|x| \geqslant 2|h|, |y| \geqslant 2|k|} |\Delta_{h,k}^{1,2} K(x, y)|\, dx\, dy \leqslant A.$$

THEOREM 1. *Suppose K is integrable on $\mathbf{R}^n \times \mathbf{R}^m$ and satisfies the properties* (C.1) *and* (S.1). *Then*

$$\|f * K\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)} \leqslant \tilde{A}\, \|f\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}$$

with the constant $\tilde{A}$ depending only on A (and not on the L^1 norm of K).

Proof. We remark that the assumptions on K are dilation invariant in the sense that $\delta_1^{-n}\delta_2^{-m}K(x/\delta_1, y/\delta_2)$ satisfies the same assumptions as K (with the same constant A), independently of $\delta_1, \delta_2 > 0$. Thus to prove the theorem which is equivalent to $|\hat{K}(\xi, \eta)| \leqslant \tilde{A}$, it suffices to show this when $|\xi| = |\eta| = 1$. To do this write $\hat{K}(\xi, \eta) = \iint K(x, y)\, e^{i(x\cdot\xi + y\cdot\eta)}\, dx\, dy = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$.

Here I is the result of integrating over the set $\{|x| > 10, |y| > 10\}$; II over the set $\{|x| \leqslant 10, |y| > 10$; III over the set $\{|x| > 10, |y| \leqslant 10\}$; and IV over the set $\{|x| \leqslant 10, |y| \leqslant 10\}$.

To estimate III we write it as

$$\iint_{|x|\geqslant 10, |y|\leqslant 10} K(x, y)[e^{iy\cdot\eta} - 1]\, e^{ix\cdot\xi}\, dx\, dy$$

$$+ \iint_{|x|\geqslant 10, |y|\leqslant 10} K(x, y) e^{ix\cdot\xi}\, dx\, dy = \mathrm{III}_1 + \mathrm{III}_2.$$

For III_1 we observe that

$$\int_{|x|\geqslant 10} K(x, y)\, e^{ix\cdot\xi}\, dx = \frac{1}{2}\int_{|x|\geqslant 10} [K(x, y) - K(x + \pi\xi, y)]\, e^{ix\cdot\xi}\, dx$$
$$+ O\left(\int_{10-\pi\leqslant |x|\leqslant 10+\pi} |K(x, y)|\, dx\right).$$

To finish the estimate for III_1 we need only insert the estimate $|e^{iy\cdot\eta} - 1| \leqslant c\,|y|$. For III_2, with $K_1(x) = \int_{|y|\leqslant 10} K(x, y)\, dy$, we have

$$\int_{|x|\geqslant 10} K_1(x)\, e^{ix\cdot\xi}\, dx = \frac{1}{2}\int_{|x|\geqslant 10} [K_1(x + \pi\xi) - K_1(x)]\, e^{ix\cdot\xi}\, dx$$
$$+ O\left(\int_{10-\pi\leqslant |x|\leqslant 10+\pi} |K_1(x)|\, dx\right).$$

Using our assumptions then leads to the desired control of III.

The estimate for II is the same as that for III, but with x and y reversed. To estimate IV we consider

$$e^{ix\cdot\xi} e^{iy\cdot\eta} = (e^{ix\cdot\xi} - 1)(e^{iy\cdot\eta} - 1) + (e^{ix\cdot\xi} - 1) + (e^{iy\cdot\eta} - 1) + 1,$$

and then our required bound follows directly from (C.1) (a) and (b) and (S.1)(a).

To estimate I we observe that

$$e^{ix\cdot\xi}e^{iy\cdot\eta}=-e^{i(x+(\pi/2)\xi)\cdot\xi}e^{i(y+(\pi/2)\eta)\cdot\eta}$$

and argue as in the case of III. This concludes the proof of the theorem.

From the theorem we shall deduce the existence of the corresponding singular integrals in the L^2 norm as a limit of truncated integrals.

If the kernel K is given we shall write

$$K_\varepsilon^N(x,y)=K(x,y) \qquad \text{if } \varepsilon_1<|x|<N_1,\ \varepsilon_2<|y|<N_2,$$
$$K_\varepsilon^N(x,y)=0 \qquad \text{otherwise,}$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)$, $N=(N_1,N_2)$.

COROLLARY. *Suppose K is locally integrable away from the cross $\{|x|=0\}\cup\{|y|=0\}$, and K satisfies* (C.1) *and* (S.1). *Then*

(a) *The operators $T_\varepsilon^N(f)=f*K_\varepsilon^N$ are bounded in L^2 with a bound independent of ε and N.*

(b) *If in addition $\iint K_\varepsilon^N(x,y)\,dx\,dy$, $\int_{\alpha_2<|y|<\beta_2}K_\varepsilon^N(x,y)\,dy$, and $\int_{\alpha_1<|x|<\beta_1}K_\varepsilon^N(x,y)\,dx$ converge (a.e.) to limits as $\varepsilon\to 0$ and $N\to\infty$, then for every $f\in L^2$, $\lim_{N\to\infty,\varepsilon\to 0}T_\varepsilon^N(f)=Tf$ exists in the $L^2(\mathbf{R}^n\times\mathbf{R}^m)$ norm and $f\to T(f)$ is bounded operator.*

Proof. Part (a) of the corollary follows from the theorem, observing that by simple calculations, K_ε^N is globally integrable and satisfies the condition (C.1) and (S.1) with an A independent of ε and N.

To prove (b), it suffices to show that $T_\varepsilon^N(f)$ converges (as $\varepsilon\to\infty$, $N\to\infty$) in the L^2 for a dense subset of $L^2(\mathbf{R}^n\times\mathbf{R}^m)$. For this purpose consider those f of the form $f_1(x)f_2(y)$, where f_i are smooth and of compact support. Assuming as we may that $\varepsilon_1,\varepsilon_2<2$ and $N_1,N_2>2$, then $T_\varepsilon^N(f)$ can be written as a sum of four terms. The first term is

$$\iint_{\varepsilon_1<|s|\leqslant 2,\varepsilon_2<|t|\leqslant 2}K(s,t)f_1(x-s)f_2(y-t)\,ds\,dt.$$

We write

$$f_1(x-s)f_2(y-t)=(f_1(x-s)-f_1(x))(f_2(y-t)-f_2(y))$$
$$+f_1(x)[f_2(y-t)-f_2(y)]+[f_1(x-s)-f_1(x)]f_2(y)+f_1(x)f_2(y).$$

Inserting this in the above we get four integrals. In view of the smoothness of f_1 and f_2 and the conditions we have assumed on K, the result of all of this is dominated by a fixed bounded function of compact support, and the limit as $\varepsilon \to 0$ and $N \to \infty$ exists for each x and y.

A second term is

$$\iint_{2 \leqslant |s| \leqslant N_1, \epsilon_2 \leqslant |t| \leqslant 2} K(s, t) f_1(x - s) f_2(y - t) \, ds \, dt.$$

This term can be easily dominated by

$$F_2(y) \int_{|t| \leqslant 2} |t|^{-m+1} \, dt \times \int_{|s| \geqslant 2} |s|^{-n} |f_1(x - s)| \, ds,$$

where F_2 is a bounded function of bounded support. Thus we obtain a domination (independent of ε and N) by a function which belongs to each $L^p(\mathbf{R}^n \times \mathbf{R}^m)$, $p > 1$; and again the limits as $\varepsilon \to 0$ and $N \to \infty$ exist a.e. The other two terms are handled in a similar way, and the proof of the corollary is concluded.

Incidentally we have shown that $T_\varepsilon^N(f)$ converges in L^p norm and almost everywhere as $\varepsilon \to 0$ and $N \to \infty$, whenever f is of the special form used above.

2. L^p Estimates

We can already obtain non-trivial L^p results from Theorem 1. One may argue as follows: Suppose, e.g., that K satisfies the cancellation and size conditions in the simplified form described in the introduction, i.e.,

$$\int_{\alpha < |x| < \beta} K(x, y) \, dx = 0 \qquad \text{for all } 0 < \alpha < \beta \text{ and } y \in \mathbf{R}^m,$$

$$\int_{\alpha < |y| < \beta} K(x, y) \, dy = 0 \qquad \text{for all } 0 < \alpha < \beta \text{ and } x \in \mathbf{R}^n,$$

and

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \, \partial y^\beta} K(x, y) \right| \leqslant A \, |x|^{-n-|\alpha|} \, |y|^{-m-|\beta|}.$$

Then if $|\alpha'| = |\alpha|$, and $|\beta'| = |\beta|$, then kernels $(\partial^{\alpha+\beta}/\partial x^\alpha \, \partial y^\beta)(x^{\alpha'} y^{\beta'} K(x, y))$ satisfy the conditions of Theorem 1. Hence

$$\left| \xi^\alpha \left(\frac{\partial}{\partial \xi}\right)^{\alpha'} \eta^\beta \left(\frac{\partial}{\partial \eta}\right)^{\beta'} \hat{K}(\xi, \eta) \right| \leqslant A$$

and so $\hat{K}$ is a multiplier satisfying a variant of the Marcinkiewicz multiplier theorem.

We prefer, however, to pursue another line of reasoning which has the advantage of requiring nearly "minimal" smoothness conditions on the kernel, and so will also be applicable in Section 4 below.

Here we impose on K the cancellation and smoothness conditions somewhat stronger than those of the previous section. We assume that there exists a fixed $\eta > 0$, so that

(a) $|\iint_{\alpha_1<|x|<\alpha_2, \beta_1<|y|<\beta_2} K(x, y)\, dx\, dy| \leqslant A.$ (C.2)

(b) If $K_1(x) = \int_{\beta_1<|y|<\beta_2} K(x, y)\, dy$ then $|K_1(x)| \leqslant A\,|x|^{-n}$, $|\Delta_h^1 K_1(x)| \leqslant A\,|h|^\eta\,|x|^{-n-\eta}$ for $|x| \geqslant 2\,|h|$, with a similar condition for $K_2(y) = \int_{\alpha_1<|x|<\alpha_2} K(x, y)\, dx$.

(a) $|K(x, y)| \leqslant A\,|x|^{-n}\,|y|^{-m}.$ (S.2)

(b) $|\Delta_h^1 K(x, y)| \leqslant A\,|h|^\eta\,|x|^{-n-\eta}\,|y|^{-m}$ if $|x| \geqslant 2\,|h|$ with a similar condition on $\Delta_k^2 K(x, y)$.

(c) $|\Delta_{h,k}^{1,2} K(x, y)| \leqslant A(|h|\,|k|)^\eta\,|x|^{-n-\eta}\,|y|^{-m-\eta}$ if $|x| \geqslant 2\,|h|$, $|y| \geqslant 2\,|k|$.

THEOREM 2. *Suppose K is integrable on $\mathbf{R}^n \times \mathbf{R}^m$ and satisfies* (C.2) *and* (S.2) *above. Then*

$$\|f * K\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leqslant A_p\,\|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)}, \qquad 1 < p < \infty,$$

with A_p depending only on A and p.

Proof. The idea of the proof is to compare appropriate square functions of f and $f * K$. Such ideas have been used before (see [11] and Calderón and Torchinsky [1] for some further variants). The situation we deal with here—that of proving estimates with minimal smoothness assumptions, and also weighted L^p estimates in the following sections—all these require a new twist to the argument.

Let $\psi^{(1)}$ be a non-trivial radial C^∞ function on $\mathbf{R}^n$ supported inside the unit ball, with $\int_{\mathbf{R}^n} \psi^{(1)}(x)\, dx = 0$, and let $\psi^{(2)}$ have similar properties on $\mathbf{R}^m$. Set $\psi(x, y) = \psi^{(1)}(x) \cdot \psi^{(2)}(y)$ and $\psi_{s,t}(x, y) = s^{-n} t^{-m} \psi(x/s, y/t)$, for $s, t > 0$. Define

$$S_\psi^2(f)(x, y) = \int_0^\infty \int_0^\infty |f * \psi_{s,t}(x, y)|^2 \frac{ds\, dt}{st}.$$

We show first that

$$A_p'\,\|f\|_{L^p} \leqslant \|S_\psi(f)\|_{L^p} \leqslant A_p\,\|f\|_{L^p}, \qquad 1 < p < \infty. \tag{2.1}$$

To do this define a function F: $\mathbf{R}^n \times \mathbf{R}^m \to L^2((0,\infty)(dt/t))$ by $F(x, y)(t) = f(x, \cdot) * \psi_t^{(2)}(y)$. Then it is clear if we define

$$S_{\psi_1}^2(F)(x, y) = \int_0^\infty |F(\cdot, y) * \psi_s^{(1)}(x)|_{L^2(t)}^2 \frac{ds}{s}$$

we have

$$S_\psi(f) \equiv S_{\psi_1}(F).$$

But if y is fixed, then S_{ψ_1} acts on $F(\cdot, y)$ in the x-variable like a classical singular integral (acting on L^2-space-valued functions). We have therefore

$$\int S_{\psi_1}^p(f)(x, y)\, dx \leqslant c \int |F|_{L^2(t)}^2(x, y)\, dx. \tag{2.2}$$

But

$$|F(x, y)|_{L^2(t)}^2 = S_{\psi_2}^2(f)(x, y) = \int_0^\infty |f * \psi_t^2(x, y)|^2 \frac{dt}{t}.$$

So integrating (2.2) in y we have

$$\iint S_\psi^p(f)\, dx\, dy \leqslant c \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} S_{\psi_2}^p(f)\, dy \right) dx.$$

But again by the standard inequalities

$$\int_{\mathbf{R}^m} S_{\psi_2}^p(f)(x, y)\, dy \leqslant c' \int_{\mathbf{R}^m} |f(x, y)|^p\, dy,$$

and intergrating in x shows that the right half of (2.1) holds. To prove the converse inequality we proceed by the usual duality argument, once we observe that,

$$\iint S_\psi(f)\, S_\psi(g)\, dx\, dy \geqslant c \left| \iint f(x, y)\, \overline{g(x, y)}\, dx\, dy \right|, \tag{2.3}$$

which follows from the identity

$$\iint S_\psi^2(f)(x, y)\, dx\, dy = c \iint |f(x, y)|^2\, dx\, dy,$$

and which in turn is a consequence of the Plancherel theorem and the fact that

$$\int_0^\infty \int_0^\infty |\hat{\psi}(\xi s, \eta t)|^2 \frac{ds}{s} \frac{dt}{t} = c', \qquad \text{all } \xi, \eta.$$

Besides using the square function S_ψ we shall also need the variant $S_{\psi*\psi}$ for which is course the same kind of results hold.

Now we can estimate $S_{\psi*\psi}(Tf)$:

$$S^2_{\psi*\psi}(Tf)(x, y) = \iint_{s,t>0} |Tf * \psi_{st} * \psi_{st}(x, y)|^2 \frac{ds\,dt}{st}$$

$$= \iint_{s,t>0} |f * \psi_{st} * K * \psi_{st}(x, y)|^2 \frac{ds\,dt}{st}.$$

We claim that $|f * \psi_{s,t} * K * \psi_{s,t}(x, y)| \leqslant CM_S(f * \psi_{st})(x, y)$ where M_S is the strong maximal operator on $\mathbf{R}^n \times \mathbf{R}^m$. In fact, if $\Phi(x, y) = 1/(1 + |x|^{n+\eta})(1 + |y|^{m+\eta})$ then (2.3) follows by a dilation argument and the estimate

$$|K * \psi(x, y)| \leqslant C\Phi(x, y). \tag{2.4}$$

To prove (2.4), we assume first that both $|x|$ and $|y| \geqslant 1$. Now

$$(K * \psi)(x, y) = \iint K(x + u, y + v)\, \psi_1(u)\, \psi_2(v)\, du\, dv,$$

and write

$$K(x + u, y + v) = \Delta^{1,2}_{u,v} K(x, y) - \Delta^1_u K)(x, y) - (\Delta^2_v K)(x, y) + K(x, y).$$

Insert this in the above and observe that the last three terms give a zero contribution to the integral, because

$$\int \psi_1(y)\, du = \int \psi_2(v)\, dv = 0.$$

Thus

$$(K * \psi)(x, y) = \iint [\Delta^{1,2}_{u,v} K(x, y)]\, \psi_1(u)\, \psi_2(v)\, du\, dv$$

so

$$|K * \psi(x, y)| \leqslant C\, |x|^{-n-\eta}\, |y|^{-m-\eta},$$

by condition (S.2)(c).

If $|x| \leqslant 2$ and $|y| \geqslant 2$ we write $(K * \psi)(x, y)$ as

$$\iint_{|u| \leqslant 3} K(u, y - v)\, \psi_1(x - u)\, \psi_2(v)\, du\, dv = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} = \iint_{|u| \leqslant 2} [K(u, y-v) - K(u, y)][\psi_1(x-u) - \psi_1(x)]\, \psi_2(v)\, du\, dv,$$

$$\mathrm{II} = \iint_{|u| \leqslant 3} [K(u, y-v) - K(u, y)]\, \psi_1(x)\, \psi_2(v)\, du\, dv,$$

and

$$\mathrm{III} = \iint_{|u| \leqslant 3} K(u, y)\, \psi_2(x-u)\, \psi_2(v)\, du\, dv.$$

Now III vanishes because $\int \psi_2(v)\, dv = 0$, and I and II are easily estimated by $C|y|^{-m-n}$. The case when $|x| \geqslant 2$, $|y| \leqslant 2$, and $|x| \leqslant 2$, $|y| \leqslant 2$ are treated similarly, thus proving (2.4).

It follows that

$$S^2_{\psi * \psi}(Tf)(x, y) \leqslant C \iint_{s,t>0} M_S^2(f * \psi_{s,t})(x, y) \frac{ds\, dt}{st}.$$

What remains is to use the norm inequality

$$\left\| \left(\iint_{s,t>0} M_S^2(f * \psi_{s,t})(x, y) \frac{ds\, dt}{st} \right)^{1/2} \right\|_{L^p} \leqslant C \left\| \left(\iint_{s,t>0} |f * \psi_{s,t}(x, y)|^2 \frac{ds\, dt}{st} \right)^{1/2} \right\|_{L^p}. \tag{2.5}$$

The right side of this last inequality is just $\|S_\psi(f)\|_{L^p} \leqslant C\, \|f\|_{L^p}$, and we are done. But (2.5) in its discrete form is just a version of the vector maximal theorem of C. Fefferman and Stein [5]:

$$\left\| \left(\sum M f_k^2 \right)^{1/2} \right\|_{L^p} \leqslant C \left\| \left(\sum f_k^2 \right)^{1/2} \right\|_{L^p}$$

after an iteration.

We now pass to the existence of the corresponding singular integrals arising by "truncation" (when we drop the assumption that K is globally integrable). As opposed to the L^2 case considered previously the nature of our problem makes it more difficult to consider truncation by sharp cut-offs. The consideration of these cut-offs (in the context of dominated convergence) will be treated later in section 4. Here we shall deal with smooth cut-offs, and for this purpose let φ_1 be a C^∞ function on $\mathbf{R}^n$ which is a radial, vanishes near $x = 0$, and is $= 1\, |x| \geqslant 1$. Let φ_2 be defined similarly. Set

$$\tilde{K}_\varepsilon^N(x, y) = K(x, y)\, \varphi_1(x/\varepsilon_1)(1 - \varphi_1(x/N_1)\, \varphi_2(y|\varepsilon_2)(1 - \varphi_2(y/N_2)).$$

COROLLARY. *Suppose K is locally integrable away from the cross $\{x=0\} \cup \{y=0\}$, and satisfies the conditions* (C.2) *and* (S.2). *Then,*

(a) *The operators $\tilde{T}_\varepsilon^N(f) = f * \tilde{K}_\varepsilon^N$ are bounded on L^p, $1 < p < \infty$, with bounds independent of ε and N.*

(b) *If in addition $\iint_{\alpha_1 < |x| < \beta_1, \alpha_2 < |y| < \beta_2} \tilde{K}_\varepsilon^N(x, y)\, dx\, dy$,*

$$\int_{\alpha_2 < |y| < \beta_2} \tilde{K}_\varepsilon^N(x, y)\, dy \quad \textit{and} \quad \int_{\alpha_1 < |x| < \beta_1} \tilde{K}_\varepsilon^N(x, y)\, dx$$

converge (a.e.) to limits as $\varepsilon \to 0$ and $N \to \infty$ then for every $f \in L^p$, $\lim_{\varepsilon \to 0, N \to \infty} \tilde{T}_\varepsilon^N f = T(f)$ exists in the L^p norm, and $f \to T(f)$ is a bounded operator on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Remark. The limiting operator T is independent of the cur-offs used to define it; and in the case $p = 2$, T agrees with the operator defined by sharp cut-offs in Section 1.

The proof of the corollary is very similar to that of the analogous corollary to Theorem 1. The main point again is that if K satisfies (C.2) and (S.2) then $\tilde{K}_\varepsilon^N$ is globally integrable and satisfies (C.2) and (S.2) with bounds independent of ε and N, as an easy calculation shows. This argument also shows that the limit is independent of the particular cut-off used.

3. THE WEIGHTED L^p CASE

We recall that a function $w(x) \geqslant 0$ defined on $\mathbf{R}^n$ satisfies the condition $A_p(\mathbf{R}^n)$, $1 < p < \infty$ if

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left\{ \frac{1}{Q} \int_Q (w(x))^{-1/(p-1)}\, dx \right\}^{p-1} \leqslant N \tag{3.1}$$

for some $N < \infty$, and all cubes Q.

As is well known, the necessary and sufficient condition that the inequality

$$\int_{\mathbf{R}^n} |T_1 f(x)|^p\, w(x)\, dx \leqslant C \int_{\mathbf{R}^n} |f(x)|^p\, w(x)\, dx \tag{3.2}$$

holds forvery standard singular integral operator T_1 is that w satisfies (3.1). For the basic facts about $A_p(\mathbf{R}^n)$ weights that we need, see the presentation in Coifman and C. Fefferman [3].

For our purposes we shall define the weight class $A_p(\mathbf{R}^n \times \mathbf{R}^m)$ to consist of all $w(x, y)$ so that for fixed $y \in \mathbf{R}^n$, $x \to w(x, y)$ is in $A_p(\mathbf{R}^n)$ and has A_p

norm (the least N in (3.1)) bounded independently of y, with a similar condition for the functions $y \to w(x, y)$, uniformly in $x \in \mathbf{R}^n$.

With this definition we are in a position to state our result.

THEOREM 3. *Suppose* $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$, *and that* K *satisfies the assumptions* (C.2) *and* (S.2) *of Theorem* 2. *Then the conclusions of Theorem* 2 *and its corollary hold if the unweighted space* L^p *is replaced throughout by* $L^p(w(x, y)\, dx\, dy)$.

We shall present the proof of this theorem by showing how the various elements of Theorem 2 extend to the weighted case.

Let us return to (3.2). Observe that if T_2 is a standard singular integral on $\mathbf{R}^m$ (acting on the y variable) then we have

$$\iint |(T_1 T_2 f)(x, y)|^p\, w(x, y)\, dx\, dy \leqslant C \int |f(x, y)|^p\, w(x, y)\, dx\, dy, \quad (3.3)$$

whenever $w(x, y) \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$.

In fact by (3.2) we have for each $y \in \mathbf{R}^m$

$$\int_{\mathbf{R}^n} |(T_1 f)(x, y)|^p\, w(x, y)\, dx \leqslant C \int_{\mathbf{R}^n} |f(x, y)|^p\, w(x, y)\, dx$$

with C independent of y. Now an integration in y shows that

$$\iint |(T_1 f)(x, y)|^p\, w(x, y)\, dx\, dy \leqslant C \int |f(x, y)|^p\, w(x, y)\, dx\, dy$$

and a repetition of this argument (with x and y intercahnged) applied to $(T_1 f)(x, y)$ gives (3.3). A similar argument also shows that

$$\iint (M_S(f)(x, y))^p\, w(x, y)\, dx\, dy \leqslant C \iint |f(x, y)|^p\, w(x, y)\, dx\, dy \quad (3.4)$$

since $M_S(f) \leqslant M_1(M_2(f))$, where M_1 and M_2 are the standard maximal functions in the x and y variables, respectively.

Next we turn to the consideration of the square functions used in the proof of Theorem 2. In fact what we observed there can be restated as

$$S_\psi(f)(x, y)) = |T_1 T_2(f)(x, y)|,$$

where T_1 is a standard (L^2 space-valued) singular integral acting on the x-variable, and similarly T_2 is a standard (L^2 space-valued) singular integral acting on the y-variable. Thus by (3.3)

$$\|S_\psi(f)\|_{L^p(w)} \leqslant C\, \|f\|_{L^p(w)}, \qquad 1 < p < \infty,$$

when $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$. The converse inequality $\|f\|_{L^p(w)} \leqslant C \|S_\psi(f)\|_{L^p(w)}$ follows as in the unweighted case from (2.3) and the observation that $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$ if and only if $w^{-1/(p-1)} \in A_q(\mathbf{R}^n \times \mathbf{R}^m)$ with $1/p + 1/q = 1$.

A final fact we need is contained in the following lemma.

LEMMA. *Suppose $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$, then*

$$\left\| \left(\sum M_S^2(f_k) \right)^{1/2} \right\|_{L^p(w)} \leqslant C \left\| \left(\sum |f_k|^2 \right)^{1/2} \right\|_{L^p(w)}.$$

Proof. As in the inequalities discussed earlier it suffices to prove the corresponding inequality for the x-variable and y-variable separately and then use the iteration argument as above. So we are reduced to showing that

$$\int_{\mathbf{R}^n} \left(\sum M_1^2(f_k)(x) \right)^{p/2} w(x)\, dx \leqslant C \int_{\mathbf{R}^n} \left(\sum |f_k(x)|^2 \right)^{p/2} w(x)\, dx \tag{3.5}$$

if $w \in A_p(\mathbf{R}^n)$, $1 < p < \infty$.

We shall show that

$$w \left\{ \left(\sum (M_1 f_k)^2 \right)^{1/2} > 10^{10^n} \alpha, M_1 \left(\left(\sum f_k^2 \right)^{1/2} \right) \leqslant \gamma \alpha \right\}$$
$$\leqslant C \gamma^\delta w \left\{ \left(\sum (M_1 f_k)^2 \right)^{1/2} > \alpha \right\} \tag{3.6}$$

for some $\delta > 0$. From this it follows that

$$\int_{\mathbf{R}^n} \left\{ \sum M_1^2(f_k) \right\}^{p/2} w\, dx \leqslant C \int_{\mathbf{R}^n} M_1^p \left(\left\{ \sum f_k^2 \right\}^{1/2} \right) w\, dx;$$

by the scalar weighted Maximal Theorem

$$\int_{\mathbf{R}^n} M_1^p \left(\left\{ \sum f_k^2 \right\}^{1/2} \right) w\, dx \leqslant C \int_{\mathbf{R}^n} \left\{ \sum f_k^2 \right\}^{p/2} w\, dx$$

if $w \in A^p$ and combining the last two inequalities, our proof of (3.5) would be complete.

So what remains is the proof of the inequality (3.6). Set

$$\Omega = \left\{ \left\{ \sum (M_1 f_k)^2 \right\}^{1/2} > \alpha \right\}.$$

Decompose Ω into Whitney cubes Q_j. We must show that

$$m\left\{x \in Q_j \;\middle|\; \left\{\sum (M_1 f_k)^2\right\}^{1/2} > 10^{10^n}\alpha, M_1\left[\left(\sum f_k^2\right)^{1/2}\right] < \gamma\alpha\right\} < C\gamma\,|Q_j|. \tag{3.7}$$

To be more precise: if $w \in A_p(\mathbf{R}^n)$ then $w \in A_\infty(\mathbf{R}^n)$, i.e.,

$$\frac{w(E \cap Q)}{w(Q)} \leqslant C\left(\frac{m(E \cap Q)}{m(Q)}\right)^{\delta},$$

for some $C > 0$, $\delta > 0$, and all cubes Q; moreover, C and δ depend only on the A_p norm of w. Thus from (3.7) follows the analogous inequality with m replaced by w (and γ by γ^δ). Adding these inequalities would prove (3.6).

To do this define $f_k^I = f_k \cdot \chi_{\tilde{Q}_j} \subset (\tilde{Q}_j$ is the dilate of Q_j be a factor of 4) and $f_k^0 = f_k - f_k^I$. The key observation is that if a function $g \geqslant 0$ lives outside $\tilde{Q}$ and $x, y \in Q \subset \mathbf{R}^n$ then $M_1(g)(x) \leqslant 5^n M_1(g)(y)$. Apply this to $M_1(f_k^0)$. We know that in the double of Q_j there is an x_0 so that $\{\sum M^2(f_k)\}^{1/2}(x_0) \leqslant \alpha$. Since f_k^0 lives outside $\tilde{Q}_k$, $Mf_k^0(x) \leqslant 5^n Mf_k^0(x_0)$, $x \in Q_j$, and so

$$\left\{\sum (Mf_k^0)^2(x)\right\}^{1/2} \leqslant 5^n\alpha, \qquad x \in Q_j.$$

As for $\{\sum (Mf_k^I\}^{1/2}$, by the vector maximal theorem on L^1,

$$m\left\{x \;\middle|\; \left\{\sum (Mf_k^I)^2\right\}^{1/2} > \alpha\right\} \leqslant \frac{C}{\alpha}\left\|\left\{\sum (f_k^I)^2\right\}^{1/2}\right\|_{L^1}.$$

And we have $1/|\tilde{Q}_k| \int_{\tilde{Q}_k} \{\sum f_k^2\}^{1/2}\,dx \leqslant \gamma\alpha$ if we assume $M(\{\sum f_k^2\}^{1/2})(x_1) < \gamma\alpha$ for some $x_1 \in Q_j$. It follows that $m\{\{\sum (Mf_k^I)^2\}^{1/2} > \alpha\} \leqslant C\gamma\,|Q_j|$ and this finishes the proof of (3.7) and with it the proof of our lemma and hence our theorem follows.

It is important to observe (for the later application in the proof of Theorem 4 below) that all our theorems hold in the case of Hilbert space-valued functions.

4. Maximal Theorems for L^p

We assume that K is locally integrable away from the cross $\{x = 0\} \cup \{y = 0\}$ and satisfies the size assumptions (S.2) described at the beginning of Section 2. Instead of the cancellation assumed there we are forced to assume the stronger condition that

$$\int_{\alpha_1<|x|<\alpha_2} K(x, y)\, dx = \int_{\beta_1<|y|<\beta_2} K(x, y)\, dy = 0,$$

$$\text{all } 0<\alpha_1<\alpha_2,\ 0<\beta_1<\beta_2.$$

We write $K_\varepsilon(x, y) = K(x, y)[1-\chi_{\varepsilon_1}(x)][1-\chi_{\varepsilon_2}(y)]$ with $\varepsilon = (\varepsilon_1, \varepsilon_2)$, and $\chi_{\varepsilon_1}(x)$ the characteristic function of the ball $|x| \leqslant \varepsilon_1$, and a similar definition for $\chi_{\varepsilon_2}(y)$.

Set $T_\varepsilon(f) = f * K_\varepsilon$.

THEOREM 4. *For* $1<p<\infty$

$$\|\sup_\varepsilon |T_\varepsilon(f)(x, y)|\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leqslant C_p \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)}. \tag{4.1}$$

Remarks. (i) The result also applies to the doubly truncated operators T^N_ε, since these can be expressed as linear combinations of the T_ε.

(ii) Since we have already observed that $\lim_{\varepsilon\to 0, n\to\infty} T^N_\varepsilon(f)(x, y)$ converges almost everywhere for a dense class of f, the results implies the existence almost everywhere of these integrals on L^p.

(iii) All these results go over without any change to $L^p(w)$, with $w \in A^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Proof. We find it convenient to make the following preliminary technical preparation. Suppose φ_1 and φ_2 are a pair of radial C^∞ functions on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, each vanishing near the origin, but $=1$ for large values of the argument. We then replace the given kernel $K(x, y)$ by a smooth truncation

$$\tilde{K}(x, y) = K(x, y)\, \varphi_1(x/\bar{\varepsilon}_1)[1-\varphi_1(x/\bar{N}_1)]\, \varphi_2(y/\bar{\varepsilon}_2)[1-\varphi_2(y/\bar{N}_2)]$$

with $\bar{\varepsilon}$ and $\bar{N}$ fixed. Observing that the estimates for $\tilde{K}$ (i.e., (C.2) and (S.2)) are independent of $\bar{\varepsilon}$ and $\bar{N}$, we prove (4.2) for $\tilde{K}$, and then at the end of the argument we let $\bar{\varepsilon}\to 0$, $\bar{N}\to\infty$, obtaining our desired result. To simplify notation we shall write K for $\tilde{K}$.

For $\alpha>-1$ set $\psi^\alpha(x) = (1-|x|^2)^\alpha \chi_{|x|\leqslant 1}(x)$ and let $\psi^{\alpha,\rho}(x) = \psi^\alpha(x/\rho)$ for $\rho>0$. Our problem here is to show that

$$\sup_{\rho_1,\rho_2>0} |f * [K(1-\psi^{0,\rho_1}(x))(1-\psi^{0,\rho_2}(y))]| \in L^p$$

if $f \in L^p$. We shall use the techniques of [13] for the theorem on maximal functions on spheres, using the right g-function, expressing ψ^β as an average of dilates of ψ^α, $\beta>\alpha$ and complex interpolation.

To begin with introduce $\varphi^{(1)}(x) \in C^\infty(\mathbf{R}^n)$ and $\varphi^{(2)}(y) \in C_c^\infty(\mathbf{R}^m)$, with

$$\int_{\mathbf{R}^n} \varphi^{(1)}(x)\,dx = \int_{\mathbf{R}^m} \varphi^{(2)}(y)\,dy = 1,$$

$$\varphi_{\rho_1}^{(1)}(x) = \rho_1^{-n}\varphi^{(1)}(x/\rho_1), \qquad \varphi_{\rho_2}^{(2)}(y) = \rho_2^{-m}\varphi^{(2)}(y/\rho_2),$$

$\varphi^{(i)} \geqslant 0$ supported inside the unit ball and let

$$\varphi_{\rho_1,\rho_2}(x, y) = \varphi_{\rho_1}^{(1)}(x) \cdot \varphi_{\rho_2}^{(2)}(y).$$

Let $\alpha > 0$. Then

$$\begin{aligned}
&(\sim) f * [K(1 - \psi^{\alpha,\rho}(x))(1 - \psi^{\alpha,\rho_2}(y))] \\
&\quad = f * [K(1 - \psi^{\alpha,\rho_2}(y) *_x \varphi_{\rho_1}^{(1)}(x)] - f * K *_y \varphi_{\rho_2}^{(2)}(y) *_x \varphi_{\rho_1}^{(1)}(x)] \\
&\qquad + \{\{[K(1 - \psi^{\alpha,\rho_1}(x))(1 - \psi^{\alpha,\rho_2}(y))] - f * [K(1 - \psi^{\alpha,\rho_2}(y)) *_x \varphi_{\rho_1}^{(1)}(x)]\} \\
&\qquad - \{f * [K *_y \varphi_{\rho_2}^{(2)}(y)(1 - \psi^{\alpha,\rho_1}(x)] - f * [K * \varphi_{\rho_2}^{(2)} *_x \varphi_{\rho_1}^{(1)}(x)]\}\} \\
&\qquad + f * [\{K *_y \varphi_{\rho_2}^{(2)}(y)\}(1 - \psi^{0,\rho_1}(x))] - f * [K * \varphi_{\rho_2}^{(2)}(y) * \varphi_{\rho_1}^{(1)}(x)] \\
&\qquad + f * K * \varphi_{\rho_1}^{(1)}(x) * \varphi_{\rho}^{(2)}(y).
\end{aligned}$$

Taken together, the terms enclosed in $\{\ \}$ are clearly dominated by $M_S(f)$ and the very last term is dominated by the non-tangential maximal function of $f * K$ (in Theorem 6 we need to use the non-tangential maximal function, while here we could just as well dominate by $M_S(f * K)\partial L^p$). All other terms are of the form

$$\sup_{\rho_1,\rho_2} |f * \{[K(1 - \psi^{\alpha,\rho_2}(y)) * \varphi_{\rho_1}^{(1)}(x)] - [K * \varphi_{\rho_2}^{(2)}(y) * \varphi_{\rho_1}^{(1)}(x)]\}|.$$

How do we handle this term? We consider the singular integral kernel H: $\mathbf{R}^n \times \mathbf{R}^m \to L^2((0, \infty)\colon d\rho_2/\rho_2)$ defined by

$$H(x, y)(\rho_2) = K(x, y)[1 - \psi^{\beta,\rho_2}(y)] - K *_y \varphi_{\rho_2}^{(2)}(x, y).$$

As long as $\beta > -1/2$ this kernel satisfies all the estimates assumed in our Theorem 2 to yield a bounded convolution operator on L^p. (We postpone the details until the end of this proof.) Now observe that

$$\psi^{\alpha,\rho_2}(y) = c_{\alpha,\beta} \int_0^1 \psi^{\beta,\rho_2/\rho}(y)\,\rho^{2\beta+2-n}(1 - \rho^2)^{\alpha-\beta-1}\,d\rho$$

and so

$$\begin{aligned}
&|(K * \varphi_{\rho_1}^{(1)})(1 - \psi^{\alpha,\rho_2}(y)) * f| \\
&\qquad \leqslant c\left(\rho_2 \int_0^{1/\rho_2} |(K * \varphi_{\rho_1}^{(1)})(1 - \psi^{\beta,1/\rho}(y)) * f|^2\,d\rho\right)^{1/2}
\end{aligned}$$

if $\alpha > \beta + 1/2$, and this is less than or equal to

$$c\left\{\int_0^\infty |(K *_x \varphi_{\rho_1}^{(1)})(1-\psi^{\beta,\rho}(y)) * f - (K * \varphi_{\rho_1}^{(1)}) * \varphi_\rho^{(2)}) * f|^2 \frac{d\rho}{\rho}\right\}^{1/2}$$
$$+ \sup_{\rho_1,\rho_2>0} |K *_x \varphi_{\rho_1}^{(1)} *_y \varphi_{\rho_2}^{(2)} * f|.$$

But the first term in braces is dominated by the vector maximal function in the x-variable applied to the vector function $H * f$. The second term is L^p since it is dominated by $M_S(K * f)$.

We have now shown that

$$\| \sup_{\rho_1,\rho_2} |f * [K(1-\psi^{\alpha,\rho_1}(x))(1-\psi^{\alpha,\rho_2}(y))]| \|_{L^p} \leqslant C_p \|f\|_{L^p} \qquad \text{if} \quad \operatorname{Re}\alpha > 0.$$

Now, in order to finish the proof we show that

$$\|\sup_\rho |f * [K(1-\psi^{\alpha,\rho}(y))]| \|_2 \leqslant C \|f\|_2$$

if $\operatorname{Re}\alpha > -1/2$. By complex interpolation this finishes the proof of Theorem 4. We, in turn, shall show this by proving that if we define an operator H by taking $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ functions into functions in $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ with values in $L^2((0, \infty); d\rho/\rho)$ defined by

$$H(f)(x, y)(\rho) = f * [K(1-\psi^{\beta,\rho}(y)) - K *_y \varphi_\rho]$$

then H is bounded for $\operatorname{Re}\beta > -1$.

To see this, let $\eta(y) \geqslant 0$ be a radial function on $\mathbf{R}^m$ which is supported in the annulus $\frac{1}{2} < |y| < 2$ and is identically 1 near $|y| = 1$. We use η to split H into two operators $H = H_1 + H_2$ where

$$H_1 f(x, y)(\rho) = f * [K(1-\eta^\rho)(1-\psi^{\beta,\rho}(y)) - K *_y \varphi_\rho](x, y)$$

and

$$H_2 f(x, y)(\rho) = f * [K\eta^\rho(1-\psi)^{\beta,\rho}](x, y).$$

Now, the point is that H_1 is a vector singular integral in the sense that the corresponding vector-valued convolution kernel satisfies all the properties of Theorem 1, (Section 1) and hence is bounded. Although this is not the case for H_2, it is not difficult to see that H_2 is bounded as well. To show this we require a simple lemma.

LEMMA. *If $J(y)$ is a Calderón–Zygmund kernel on $\mathbf{R}^m$, i.e., if*

(a) $\int_{\alpha<|y|<\beta} J(y)\,dy = 0$ *for all* $\alpha<\beta$,

(b) $|J(y)| \leqslant \dfrac{C}{|y|^m}$,

and

(c) $|J(y+k)-J(y)| \leqslant C\dfrac{|k|^\gamma}{|y|^{m+\gamma}}$, $\quad 2|k|<|y|$,

and if $\operatorname{Re}\beta > -1$ *then the operator* $\tilde{J}$ *defined by*

$$\tilde{J}f(y)(\rho) = f * [J\eta^\rho(1-\psi^{\beta,\rho}(y))]$$

is a bounded operator from $L^2(\mathbf{R}^m)$ *to the space of all* L^2 *functions on* $\mathbf{R}^m$ *with values in the Hilbert space* $L^2[(0,\infty); d\rho/\rho]$.

Proof. This is a simple Fourier transform argument. By the Plancherel theorem

$$\|\tilde{J}f\|^2 = \int_{\mathbf{R}^m} |\tilde{J}f\hat{\ }(\xi)|^2_{L^2(d\rho/\rho)}\,d\xi$$
$$= \int_{\mathbf{R}^m}\int_0^\infty |\hat{f}(\xi)|^2\,|J\eta(1-\psi^{\beta,\rho})]\hat{\ }(\xi)|^2\,\frac{d\rho}{\rho}\,d\xi$$

and this will be bounded by $C\int_{\mathbf{R}^m}|\hat{f}(\xi)|^2\,d\xi = \|f\|_2^2$ provided we show that

$$\int_0^\infty |[J\eta^\rho(1-\psi^{\beta,\rho})]\hat{\ }(\xi)|^2\,\frac{d\rho}{\rho}$$

is bounded as a function of ξ. This is clear, since by the dilation invariance of the estimates on J we see that $|J\eta^\rho(1-\psi^{\beta,\rho})\hat{\ }(\xi)| \leqslant Q(\rho\,|\xi|)$, where $Q(\xi)$ is a function satisfying $|\hat{p}(\xi)| \leqslant Q(|\xi|)$ for all functions $p(y)$ supported in the annulus $\frac{1}{2}<|y|<2$ satisfying an L^1 Lipschitz condition

$$\|p\|_{\Lambda^1_\delta} \leqslant C \quad \text{(here } \delta \text{ and } C \text{ are fixed)}$$

and satisfying $\int_{\mathbf{R}^m} p(y)\,dy = 0$. Furthermore, it is clear that we may choose such a majorant Q to satisfy $|Q(|\xi|)| \leqslant C'\,|\xi|^{-\delta}$ for $|\xi|$ large and $|Q(|\xi|)| \leqslant C''\,|\xi|$ for $|\xi|$ small, so that the convergence of the integral $\int_0^\infty |Q(\rho)|^2(d\rho/\rho)$ is assured. Now, to show the L^2 boundedness of H_2 we argue from the lemma as follows. For each $x\in\mathbf{R}^n$ consider the Calderón–Zygmund operator $\mathcal{O}_x$ which is convolution (in the y variable) with the kernel $K(x, \cdot)$. Then by the assumptions on the kernel K the map $x \to \mathcal{O}_x$ is an operator valued Calderón–Zygmund kernel in the x-variable. The lemma tells us that the map $x \to^{\tilde{\mathcal{O}}} \mathcal{O}_x$ is also an operator valued Calderón–Zygmund kernel in the x-variable.

Now if $f(x, y)$ is an L^2 function on $\mathbf{R}^n \times \mathbf{R}^m$ we let $f_x(y) = f(x, y)$. The map $x \to^F f_x$ is then an L^2 valued function of the x-variable. We have that $\|F * \tilde{\mathcal{O}}\|_{L^2} \leqslant C' \|F\|_{L^2}$, since the classical Calderón–Zygmund theory extends to Hilbert space operator valued kernels (see [11]). To finish things, we have only to notice that for $f \in L^2(\mathbf{R}^n \times \mathbf{R}^m)$, $\|f\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)} = \|F\|_{L^2(\mathbf{R}^n)}$, and that $F * \tilde{\mathcal{O}}(x) = (H_2 f)_x$. Finally, we shall now show that the kernel

$$H = H(x, y, \rho_2) = K(x, y)(1 - \psi^{\alpha, \rho_2}(y)) - K *_y \varphi_{\rho_2}^{(2)}(x, y)$$

satisfies the conditions (C.2) and (S.2) for functions which take their values in the Hilbert space $L^2((0, \infty); d\rho_2/\rho_2)$ and this will complete the proof of Theorem 4.

We begin by making the following easily verified observations about $K(x, y)$:

$$|K *_y \varphi_{\rho_2}^{(2)}(x, y)| \leqslant \frac{C}{|x|^n} \frac{1}{(\rho_2 + |y|)^m}, \tag{4.1}$$

$$|K(x, y) - K *_y \varphi_{\rho_2}^{(2)}(x, y)| \leqslant \frac{C}{|x|^n} \frac{\rho_2^{\eta_2}}{|y|^{m+\eta_2}} \tag{4.2}$$

for $|y| \geqslant c\rho_2$, with some $\eta_2 > 0$.

Now write $1 \equiv \xi_1(y) + \xi_2(y) + \xi_3(y)$, where ξ_1 is supported in $0 \leqslant |y| \leqslant \frac{3}{4}$, ξ_2 is supported in $\frac{1}{2} \leqslant |y| \leqslant 2$, and ξ_3 is supported in $|y| \geqslant \frac{3}{2}$, and all ξ_i are assumed to be smooth. Write

$$H = H_1 + H_2 + H_3 = H\xi_1(y/\rho_2) + H\xi_2(y/\rho_2) + H_2\xi(y/\rho_2).$$

To estimate $|H_3(x, y)|_{L^2((0,\infty); d\rho_2/\rho_2)}$, observe that by (4.2) it is easily seen to be dominated by

$$C/|x|^n \left(\int_0^{C|y|} \rho_2^{2\eta_2} \frac{d\rho_2}{\rho_2}\right)^{1/2} \frac{1}{|y|^{m+\eta_2}},$$

so $|H_3(x, y)| \leqslant C|x|^{-n} |y|^{-m}$. Similarly, suing (4.1), we get $|H_1(x, y)| \leqslant C|x|^{-n} |y|^{-m}$. The estimate for H_2 reduces essentially to

$$|x|^{-n} \left(\int_{|y|/2}^{2|y|} (1 - |y|^2/\rho_2^2)^{2\alpha} \frac{d\rho}{\rho}\right)^{1/2} |y|^{-m},$$

which is $\leqslant C|x|^{-n}|y|^{-m}$, if $\alpha > -\frac{1}{2}$. Combining the above we have $|H(x, y)| \leqslant C|x|^{-n}|y|^{-m}$. The proof of the other properties required for H (i.e., (S.2) and (C.2)) are analogous. To show, e.g., that $|\Delta_k^2 H(x, y)| \leqslant C|x|^{-n}|k|^{\eta_2'}|y|^{-m-\eta_2'}$ for $|y| \geqslant 2|k|$, and some $\eta_2' > 0$ we replace (4.1) and (4.2), respectively, by

$$|\Delta_k^2 K *_y \varphi_{\rho 2}^{(2)}(x, y)| \leqslant \frac{C}{|x|^n} \frac{|k|}{\rho_2} \frac{1}{(\rho_2 + |y|)^m} \tag{4.1$'$}$$

and

$$|\Delta_k^2[K(x, y) - K *_y \varphi_{\rho_2}^{(2)}(x, y)]| \leqslant \frac{C}{|x|^n} |k|^{\eta_2'} |\rho|^{\eta_2''} |y|^{-m-\eta_2'-\eta_2''} \tag{4.2$'$}$$

for $|y| \geqslant 2|k|$, $|y| > c\rho_2$, and for some $\eta_1' > 0$, $\eta_2'' > 0$. The other conditions are obtained similarly.

5. Maximal Theorems for $L \log^+ L$

In this section we shall deal with maximal inequalities for truncated singular integrals acting on functions in the class $L \log^+ L$. Here $K(x, y)$ will be a kernel with support contained in $\{|x| < 1\} \times \{|y| < 1\} = Q_1$. K will be C^∞ away from the cross $\{x = 0\} \cup \{y = 0\}$ and will satisfy

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leqslant C_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-m-|\beta|} \qquad \text{for all } \alpha, \beta. \tag{1}$$

$$\int_{a<|x|<b} K(x, y)\, dx = 0 \qquad \text{for all } 0 < a < b, \text{ and } y \in \mathbf{R}^m. \tag{2}$$

$$\int_{a<|y|<b} K(x, y)\, dy = 0 \qquad \text{for all } 0 < a < b, \text{ and } x \in \mathbf{R}^n.$$

We shall work with smooth truncations in the theorem below. These are defined by $K_{\varepsilon_1 \varepsilon_2}(x, y) = K(x, y)[1 - \psi_1(x/\varepsilon_1)][1 - \psi_2(y/\varepsilon_2)]$, where ψ_1 and ψ_2 are smooth radial functions on $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, each equaling 1 near the origin, and vanishing for sufficiently large values of their argument.

Theorem 5. *Let $f \in L \log^+ L(Q_1)$ and K be as above. Let $T_\varepsilon f = f * K_\varepsilon$, where K_ε is a smooth truncation of K, and $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Let*

$$T^* f(x, y) = \sup_{\varepsilon_i > 0} |T_\varepsilon f(x, y)|.$$

Then

$$m\{(x, y) \in Q_1 \mid T^* f > \alpha\} \leqslant \frac{C}{\alpha} \|f\|_{L \log^+ L(Q_1)}.$$

Proof. Let us consider the function f on $Q_2 = \{|x| < 2\} \times \{|y| < 2\}$ and write

$$f = \tilde{f} + f_1 + f_2, \qquad \text{where} \quad \tilde{f}, f_1 \in L \log^+ L(Q_2),$$

$$\int_{|x|<2} \tilde{f}(x, y)\, dx = 0, \qquad y \in \mathbf{R}^m$$

$$\int_{|y|<2} \tilde{f}(x, y)\, dy = 0,$$

$x \in \mathbf{R}^n$ and f_1 depends only on x, f_2 depends only on y. Then for $(x, y) \in Q_1$, $T_\varepsilon f = T_\varepsilon \tilde{f}$. Therefore it suffices to show that $m\{T^*\tilde{f} > \alpha\} \leqslant (C/\alpha) \|f\|_{L\log^+ L}$. Now an elementary computation using the maximal definition shows that $\tilde{f} \in H^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $p < 1$. (For the H^p theory on product domains used below, see Gundy and Stein [7] and Merryfield [9].) We now proceed to start the proof as in the L^p case writing out $\tilde{f} * [K(1 - \psi_1(x/\varepsilon_1))(1 - \psi_2(y/\varepsilon_2))]$ exactly as in $(\sim)$. Again the terms in braces are controlled by $M_S(\tilde{f})$ and since M_S maps $L \log^+ L$ boundedly to weak L^1 these terms are harmless. The term $\tilde{f} * K *_x \varphi_{\varepsilon_1} *_y \varphi_{\varepsilon_2}$ is controlled by the nontangential maximal function of $\tilde{f} * K$. Now the proof of Theorem 2 shows that if $\tilde{f} \in H^p(\mathbf{R}^n \times \mathbf{R}^m)$ then $\tilde{f} * K$ will have its square function in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Using the methods of Gundy and Stein [7] (where $n = m = 1$) we can show that the nontangential maximal function of $\tilde{f} * K \in L^p$ (if $p < 1$ is close to 1) with $\|(\tilde{f} * K)^*\|_{L^p} \leqslant C \|\tilde{f}\|_{H^p} \leqslant C \|f\|_{L\log^+ L}$. The other terms in $(\sim)$ are controlled by defining vector-valued kernels like

$$H(x, y)(\rho) = K(x, y)(1 - \eta(y/\rho)) - K *_y \varphi_\rho].$$

Here η is chosen so that for $r > 0$

$$\psi_2(ry) = \frac{1}{r} \int_0^r \eta(ty)\, dt.$$

Notice that since ψ_2 is radial this determines η by

$$\eta(r, \mathbf{1}) = \frac{\partial}{\partial r} (r\psi_2(r\mathbf{1})),$$

where $\mathbf{1}$ is any unit vector; as a result η is C^∞ and radial, is 1 near the origin and vanishes for large values of the argument. For the kernel $H(x, y)(\rho)$, with values in $L^2((0, \infty); d\rho/\rho)$, we can show as before that $\tilde{f} * H \in H^p(\mathbf{R}^n \times \mathbf{R}^m)$ with values in $L^2((0, \infty); d\rho/\rho)$. Now we invoke the fact that if the square function of a Hilbert space valued function on $\mathbf{R}^n \times \mathbf{R}^m$ belongs to L^p then so does the nontangential maximal function (again the extension of the scalar case when $n = m = 1$ is rather routine) and we have

$$\| \sup_{\rho_1 > 0} |\varphi_{\rho_1} *_x \tilde{f} * H|_{L^2[(0,\infty);d\rho/\rho]} \|_{L^p} \leqslant C \|f\|_{L \log^+ L}.$$

Proceeding as in the L^p case this yields

$$m\{ \sup_{\rho_1, \rho_2 > 0} |f * [K(1 - \psi_2(y/\rho_2) *_x \varphi_{\rho_1}| > \alpha\} \leqslant \frac{C}{\alpha^p} \|f\|^p_{L \log^+ L},$$

since

$$f * [K(1 - \psi_2(y/\rho_2)] = \frac{1}{\rho_2} \int_0^{\rho_2} f * [K(1 - \eta(y/\varepsilon_2)] \, d\varepsilon_2.$$

The theorem now follows from the following non-periodic version of the theorem on limits of sequences of operators [12]: Let $f \in L \log^+ L(Q_1)$.

Suppose $T_n f = f * K_n$, where K_n are kernels with uniformly bounded support. Assume $\sup_n |f * K_n| (x, y) < \infty$ a.e. for all $f \in L \log^+ L(Q_1)$. Let $Q_1 \subset \tilde{Q}_1$. Then

$$m\{(x, y) \in \tilde{Q}_1 \,|\, \sup_n |T_n f(x, y)| > \alpha\} \leqslant \frac{C}{\alpha} \|f\|_{L \log^+ L(Q_1)}.$$

This is in turn an easy consequence of the periodic case.

We shall now give a simple proof of the version of Theorem 5 where the kernel $K(x, y)$ is a product of the form $K_1(x) \cdot K_2(y)$ and where now

$$K_{\varepsilon_1, \varepsilon_2}(x, y) = K(x, y)(1 - \chi_{|x| < \varepsilon_1}(x))(1 - \chi_{|y| < \varepsilon_2}(y)).$$

THEOREM 6 (See C. Fefferman [4]). *Let $K_1(x)$ and $K_2(y)$ be Calderón–Zygmund kernels on $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. (Here we mean that*

$$|K_1(x)| \leqslant \frac{C}{|x|^n}, \qquad \left| \int_{\alpha < |x| < \beta} K_1(x) \, dx \right| \leqslant C,$$

and

$$|K_1(x + h) - K_1(x)| \leqslant \frac{C |h|^\eta}{|x|^{n+\eta}}$$

for $|h| \leqslant |x|/2$, and similar assumptions on $K_2(y)$). Let $f \in L \log^+ L(Q_1)$ and let

$$T_\varepsilon f(x, y) = f * K_{\varepsilon_1, \varepsilon_2}(x, y).$$

Then

$$m\{(x,y)\in Q_1 \mid \sup_{\varepsilon} |T_\varepsilon f(x,y)| > \alpha\} \leqslant \frac{C'}{\alpha} \|f\|_{L\log^+ L}.$$

We shall require a preliminary lemma.

LEMMA. *If $K(y)$, $y \in \mathbf{R}^m$ is a Calderón–Zygmund kernel, then*

$$\|f *_y K\|_{H^p(\mathbf{R}^n\times\mathbf{R}^m)} \leqslant C\,\|f\|_{H^p(\mathbf{R}^n\times\mathbf{R}^m)}.$$

(*Here $*_y$ denotes convolution in the y variable only.*)

Proof of Lemma. Let us compute the square function of $f *_y K$:

$$S_\psi^2(f *_y K)(x,y) = \iint_{\Gamma_1\times\Gamma_2} |f *_y K *_x \psi^1_{\rho_1} *_y \psi^2_{\rho_2}(x-s, y-t)|^2\, ds\, dt\, \frac{d\rho_1\, d\rho_2}{\rho_1^{n+1}\rho_2^{m+1}};$$

(here $\psi(x,y)=\psi^1(x)\,\psi^2(y)$ and $\psi^1 \in C_c^\infty(\mathbf{R}^n)$, $\psi^2 \in C_c^\infty(\mathbf{R}^m)$ with $\int \psi_i = 0$).

Now if we define a vector $(L^2(\Gamma_1;\ ds\, d\rho_1/\rho_1^{1-n}))$ valued function on $\mathbf{R}^n \times \mathbf{R}^m$, F, given by

$$F(x,y)(s,\rho) = f *_x \psi^1_{\rho_1}(x-s, y)$$

then

$$S^{(y)}_{\psi^2}(F *_y K)(x,y) = S_\psi(f *_y K),$$

where $S^{(y)}_{\psi^2}$ denotes the square function in the y variable. Then

$$\int_{\mathbf{R}^n} S_\psi^p(f *_y K)(x,y)\, dy = \int_{\mathbf{R}^m} S^{(y)p}_{\psi^2}(F *_y K)(x,y)\, dy \leqslant C\,\|(F *_y K)_x\|^p_{H^p(\mathbf{R}^m)}.$$

But convolution on $\mathbf{R}^m$ with K preserves vector valued H^p so this is

$$\leqslant C'\,\|F_x\|^p_{H^p(\mathbf{R}^m)}$$

$$\leqslant C'' \int_{\mathbf{R}^m} S^{(y)p}_{\psi^2}(F)(x,y)\, dy.$$

But $S^{(y)}_{\psi^2}(F) \equiv S_\psi(f)$ and so $\int_{\mathbf{R}^m} S_\psi^p(f *_y K)(x,y)\,dy \leqslant C'' \int_{\mathbf{R}^m} S_\psi^p(f)(x,y)\,dy$. Integrating in x completes the proof of the lemma.

Proof of the Theorem. As in Theorem 5 we may assume

$\int_{\mathbf{R}^m} f(x, y)\, dy = 0$ for all $x \in \mathbf{R}^n$, $\int_{\mathbf{R}^n} f(x, y)\, dx = 0$ for all $y \in \mathbf{R}^m$ so that $f \in H^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $p < 1$. Then it is easy to see that

$$\sup_{\varepsilon_1, \varepsilon_2} |f * K_{\varepsilon_1, \varepsilon_2}| \leqslant M_x(\sup_{\varepsilon_2} |f *_y K_{2,\varepsilon_2}|) + M_y(\sup_{\varepsilon_2} |f *_x K_{1,\varepsilon_1}|)$$
$$+ M_x M_y f + \sup_{\varepsilon_1, \varepsilon_2} | \varphi_{\varepsilon_1} *_x \varphi_{\varepsilon_2} *_y f * K|.$$

The first three terms are clearly in weak L^1, and the last term is dominated by the nontangential maximal function of an H^p function and so is in L^p, $p < 1$. The proof is now concluded by applying the limits of sequences of operator theorems just as in Theorem 5.

6. Some Examples

The class of examples of singular integrals we shall discuss in some detail are those that are suggested by the study of the operators arising in some boundary-value problems and the $\bar{\partial}$-Neumann problem in particular. (See Phong and Stein [10, Sect. 4].) A very particular example is as follows. We consider $\mathbf{R}^n \times \mathbf{R}^1$ (i.e., $m = 1$) and let

$$K(x, y) = \frac{x_k}{(|x|^2 + y^2)^{(n+1)/2}} \cdot \frac{1}{|x|^2 + iy}.$$

This kernel is a product of kernels each of which is homogeneous with differing types of homogeneity. Let us consider this situation more generally.

Assume that we are given two sets of (one-parameter) dilations on $\mathbf{R}^n \times \mathbf{R}^m$. The first is

$$(x, y) \to (\delta x, \delta^a y), \qquad \text{all } \delta > 0, \tag{6.1}$$

for some fixed $a > 0$. The second is

$$(x, y) \to (\delta^b x, \delta y), \qquad \text{all } \delta > 0. \tag{6.2}$$

for some fixed $b > 0$.

We consider K on $\mathbf{R}^n \times \mathbf{R}^m$ given by $K(x, y) = K^1(x, y) \cdot K^2(x, y)$, where K^1 is homogeneous of degree $-n$ with respect to (6.1), and K^2 is homogeneous of degree $-m$ with respect to (6.2). Both K^1 and K^2 are assumed smooth outside the origin.[1] In addition to this, some cancellation conditions must be required. The simplest are

$$K^1(x, 0) \text{ has mean-value zero on the unit sphere of } \mathbf{R}^n. \tag{6.3}$$

[1] A close examination of the arguments below shows that one merely needs to require Hölder continuity of both K^1 and K^2 away from the origin.

Similarly

$$K^2(0, y) \text{ has mean-value zero on the unit sphere of } \mathbf{R}^m. \tag{6.3'}$$

A further cancellation condition must be imposed which does not follow from (6.3) and (6.3′), namely,

$$\left| \iint_{\alpha_1 < |x| < \alpha_2, \beta_1 < |y| < \beta_2} K(x, y)\, dx\, dy \right| \leqslant A, \qquad \text{all } 0 < \alpha_1 < \alpha_2,\ 0 < \beta_1 < \beta_2. \tag{6.4}$$

(observe that for the examples described above these conditions are easily verified; here $a = 1$, $b = \frac{1}{2}$.)

PROPOSITION. *Assume that the kernel* K *satisfies all the conditions described above. Then it satisfies the hypotheses and hence the conclusion of the corollaries of Theorems* 1, 2, *and* 3.

Proof. A simple argument of homogeneity (with respect to (6.1)) shows that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} K^1(x, y) \right| \leqslant A_{\alpha,\beta}(|x| + |y|^{1/a})^{-n-|\alpha|-a|\beta|}.$$

Therefore

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} K^1(x, y) \right| \leqslant A_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-|\beta|}.$$

Similarly, using the homogeneity (6.2), we get

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} K^2(x, y) \right| \leqslant A_{\alpha,\beta} |x|^{-|\alpha|} |y|^{-m-|\beta|}.$$

Therefore by Leibniz's rule

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} K(x, y) \right| \leqslant A_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-m-|\beta|}. \tag{6.5}$$

Next suppose $K_2(y) = \int_{\alpha_1 < |x| < \alpha_2} K(x, y)\, dy$. We want to show that

$$\text{(a)} \quad |K_2(y)| \leqslant A\, |y|^{-m}, \tag{6.6}$$

$$\text{(b)} \quad |\nabla_y K_2(y)| \leqslant A\, |y|^{-m-1}.$$

We consider the case when $|y| \leqslant 1$, and make the assumption that $1/a \geqslant b$. (Reversing the inequality $|y| \leqslant 1$ and/or the inequality $1/a \geqslant b$ leads to parallel estimates.) Under the assumptions we have just made, $0 < |y|^{1/a} \leqslant |y|^b \leqslant 1$. Now write $K_2(y)$ as a sum of three integrals, taken over the sets $S_1, S_2, S_3 \subseteq \mathbf{R}^n$. Here

$$S_1 = \{\alpha_1 \leqslant |x| \leqslant |y|^{1/a}\},$$
$$S_2 = \{\alpha_1 \leqslant |x| \leqslant \alpha_2\}, \qquad \text{with} \quad |y|^{1/a} \leqslant \alpha_1 \leqslant \alpha_2 \leqslant |y|^b,$$

and

$$S_3 = \{|y|^b \leqslant |x| \leqslant \alpha_2\}.$$

(One or more of these sets may actually be empty.) For S_1 we use the fact that $|K^1| \leqslant A\,|y|^{-n/a}$ and $|K^2| \leqslant A\,|y|^{-m}$. Thus

$$\int_{S_2} |K| \leqslant A \left(A \int_{|x| \leqslant |y|^{1/a}} dx\right) |y|^{-m} = A'\,|y|^{-m}.$$

For S_3 we use the estimate $|K^1| \leqslant A\,|x|^{-n}$, $|K^2| \leqslant A\,|x|^{-m/b}$. So

$$\int_{S_3} |K| \leqslant A \int_{|x| \geqslant |y|^b} |x|^{-n}\,|x|^{-m/b}\,dx = A'\,|y|^{-m}.$$

For S_2, since $|x| \geqslant |y|^{1/a}$, then we can write (using homogeneity) that

$$K^1(x, y) = K^1(x, 0) + O\left(\frac{|y|}{|x|^{n+a}}\right).$$

Similarly, since $|x| \leqslant |y|^b$, we have

$$K^2(x, y) = K^2(0, y) + O\left(\frac{|x|}{|y|^{m+b}}\right).$$

Hence, on S_2

$$K = K^1K^2 = K^1(x, 0)\,K^2(0, y) + O(|x|^{-n+1}\,|y|^{-m-b}) + O(|y|^{-m+1}\,|x|^{-n-a}).$$

To compute $\int_{S_2} K$ we note that first term contributes 0, the second is bounded by

$$|y|^{-m-b} \int_{|x| \leqslant |y|^b} |x|^{-n+1}\,dx = A\,|y|^{-m},$$

and the third by

$$|y|^{-m+1} \int_{|x| \geqslant |y|} |x|^{-n-a} \, dx = A \, |y|^{-m}.$$

This proves (6.6)(a). The proof of (6.6)(b) is similar. It is not hard to see that (6.4), (6.5), and (6.6) (and its analogue with x replace by y) imply all the cancellation and size conditions required by the hypotheses for Theorems 1, 2, and 3.

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